Mathematics of Data: From Theory to Computation

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Supplementary Material: Asymptotics

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Asymptotic Notation

What is this notation?

- Asymptotic Notation (Big-Oh Notation, Landau's notation) describes asymptotic growth of functions.
- It is usually used to describe:
  - Running time of an algorithm
  - Memory storage require by an algorithm
  - Error achieved by an approximation
- Exact computation of the running time, memory, or error is usually not important: For large inputs, multiplicative constants and lower-order terms "do not matter."

Examples

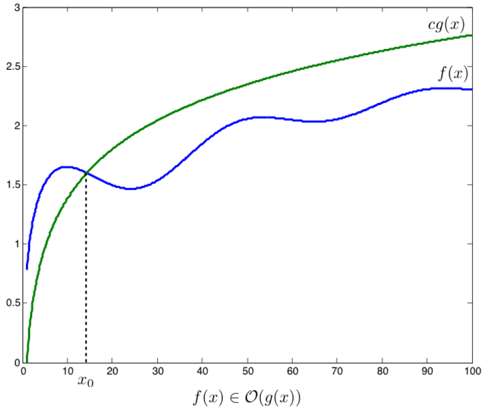
- Binary search's running time in a sorted list of n elements. [1]: $O(\log(n))$
- Number of comparisons required for sorting a list of n elements [1]: $\Omega(n \log(n))$

Asymptotic Notation: Big-Oh

Definition (Big-Oh)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in O(g(x)) \text{ iff } \exists c > 0, \exists x_0, \text{such that } |f(x)| \leq c|g(x)|, \forall x \geq x_0$$



- In computer science, the definition is taken over positive integers.

Example

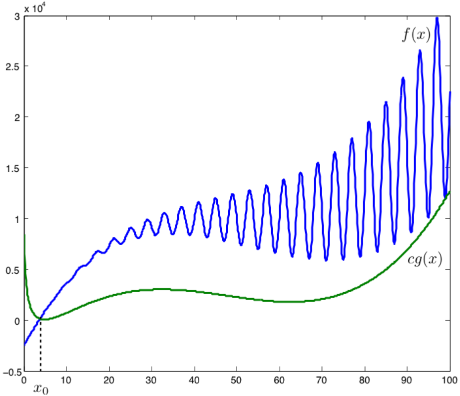
- $x \in O(x^2)$
- $\log(n!) \in O(n \log(n))$
- $n^{1+\sin(n)} \in O(n^2)$

Asymptotic Notation: Big-Omega

Definition (Big-Omega)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in \Omega(g(x)) \text{ iff } \exists c > 0, \exists x_0, \text{such that } |f(x)| \geq c|g(x)|, \forall x \geq x_0$$



$f(x) \in \Omega(g(x))$

- Intuition: g is a lower bound of f iff f is an upper bound of g.
- $f(x) \in \Omega(g(x)) \Leftrightarrow g(x) \in O(f(x))$.

Example

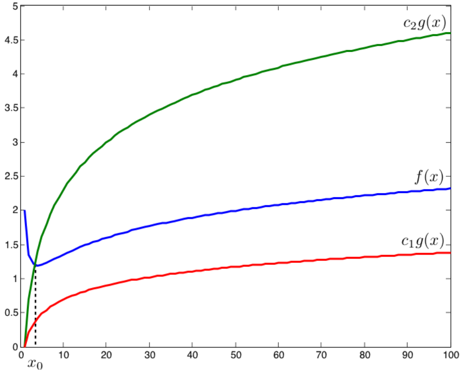
- $x^2 \in \Omega(x)$
- $\log(n!) \in \Omega(n \log(n))$
- $n^{1+\sin(n)} \in \Omega(1)$

Asymptotic Notation: Theta

Definition (Theta)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in \Theta(g(x)) \text{ iff } \exists c_1, c_2 > 0, \exists x_0, \text{such that } c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2, \forall x \geq x_0$$



$f(x) \in \Theta(g(x))$

- **Intuition:** g is a tight bound for f iff it is both an upper and a lower bound of it.
- $f(x) \in \Theta(g(x))$ iff $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$.
- $f(x) \in \Theta(g(x))$ iff $g(x) \in \Theta(f(x))$.

Example

- $\sin(x) \in \Theta(1)$
- $x + \log(x) \in \Theta(x)$
- $\log(n!) \in \Theta(n \log(n))$
- **Stirling's approximation:** $n! \in \sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))$

Asymptotic Notation: small-oh and small-omega

Definition (small-oh, small-omega)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in o(g(x)) \text{ iff } \forall c > 0, \exists x_0, \text{such that } |f(x)| \leq c|g(x)|, \forall x \geq x_0,$$

or equivalently $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = 0$.

$$f(x) \in \omega(g(x)) \text{ iff } \forall c > 0, \exists x_0, \text{such that } |f(x)| \geq c|g(x)|, \forall x \geq x_0,$$

or equivalently $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \infty$.

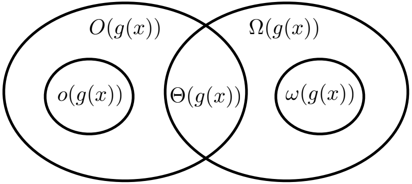
- These are non-tight upper/lower bounds.
- $g(x) \in o(f(x))$: g is dominated by f asymptotically.
- $f(x) \in \omega(g(x))$: f dominates g asymptotically.
- $f(x) \in \omega(g(x)) \Leftrightarrow g(x) \in o(f(x))$.

Example

- $\frac{1}{x} \in o(1)$
- $5 \in \omega(\frac{1}{x})$
- $n! \in o(n^n)$
- $n! \in \omega(2^n)$

Hierarchy of asymptotic notation classes

- Relation between the different asymptotic notations:



- Analogy with real numbers comparison:

Aymptotic function comparison	Real numbers comparison
$f(x) = O(g(x))$	$a \leq b$
$f(x) = \Omega(g(x))$	$a \geq b$
$f(x) = \Theta(g(x))$	$a = b$
$f(x) = o(g(x))$	$a < b$
$f(x) = \omega(g(x))$	$a > b$

- Difference from real numbers comparison: Not all functions are **asymptotically comparable**, e.g., n, $n^{1+\sin(n)}$.

Asymptotic Notation: some remarks

Some notation abuse:

- Use of equality: $f(x) = O(g(x))$

Some variations:

- Soft-Oh: $\tilde{O}(\cdot)$ notation ignores log terms, i.e., $O(x^c \log^k(x)) = \tilde{O}(x^c)$.
- Asymptotic notation can also describe limiting behavior as $x \to a$, e.g., $e^x = 1 + x + \frac{x^2}{2} + o(x^2), x \to 0$ (by Taylor's theorem).

References I

[1] T. Cormen, C. Leiserson, R. Rivest, C. Stein, et al.
Introduction to algorithms, volume 2.
MIT press Cambridge, 2001.
(Cited on page 3.)